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On the Construction of Matter Tensors in Crystals

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Matter tensors, complying with the symmetry of a crystal, may be found by application of Wigner's theorem. For higher-order tensors a modification of Fumi's method is useful for all classes. The trigonal and hexagonal classes present no special difficulty. The two methods are illustrated on a sixth-order tensor.

1. Introduction

The derivation of tensors in the several crystal classes has been considered by many authors. The simplest method is the direct inspection method, which is used in the textbook of Nye (1957; *cf.* Fumi, 1952*a*, *b*). The application to trigonal and hexagonal classes, however, leads to rather intricate calculations. Fumi (1952*a*; *cf.* Fieschi & Fumi, 1953) has also devised a group-theoretical method for these two crystal systems. Only the latter method will be referred to here as the Fumi method, the term 'direct inspection method' being reserved for the former.

It is possible to modify the direct inspection method in such a way that trigonal and hexagonal crystals can be treated in the same way as others, at least for second-, third- and fourth-order tensors. Fumi's method is more powerful for higher-order tensors. But here it is not necessary to study each crystal class in detail, as he has done. It will be shown that quite simple calculations define the tensors for all classes at the same time.

This paper will not deal with detailed group-theoretical studies; thus the papers by Jahn (1937, 1949) and Bhagavantam (1952) will not be used. For the sake of completeness, attention is drawn to the work of Sirotin (1960), who builds the tensors from a collection of so-called basic tensors.

I. THE MODIFIED DIRECT METHOD

2. Basic concepts

It is well known that tensor components transform as products of coordinates. Let us denote temporarily the coordinates x, y, z by x_i, $i=1, 2, 3$; then a transformation of space (*e.g.* a rotation) is given by

$$x'_i = u_{ik}(s)x_k$$

where s is the symbol for the transformation we consider. Now the transformation of a tensor component t_{ijk} is the same as the transformation of the product $x_i x_{1,j} x_{2,k}$ of the coordinates of three vectors $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2$.

$$t'_{ijk} = u_{ip}(s)u_{jq}(s)u_{kr}(s)t_{pqr}\ .$$

We may denote this transformed component by $t_{ijk}(s)$.

A matter tensor in a crystal must be invariant with respect to all symmetry operations s of the class we consider. Let $\mathbf{T}$ be an invariant tensor. The direct method puts

$$T_{ijk}(s) = T_{ijk}$$

for all operations. This method is in general very simple, but it is rather unwieldy for trigonal and hexagonal classes. Let us take as an example the rotation of 180° around the Z axis:

$$x'_1 = -x_1,\ x'_2 = -x_2,\ x'_3 = x_3\ .$$

It follows that

$$T_{112} = -T_{112} = 0 \; .$$

We may start, however, with an arbitrary tensor $\mathbf{t}$. From this tensor we construct an invariant tensor $\mathbf{T}$ by using Wigner's theorem (Fumi, 1952; Wigner, 1931). We find

$$T_{ijk} = \sum_s t_{ijk}(s) \qquad (1)$$

summed over all operations of the class, including the identity. For, as one of these transformations is applied to $\mathbf{T}$, each term of the sum shifts to another term, and the sum remains the same. Taking the same example as above, we find

$$T_{112} = t_{112} - t_{112} = 0 \; ,$$

the first term at the right hand belonging to the identity, the second to the rotation.

Now for a group of high order the summation (1) would be very tedious. But it is sufficient to study the cyclic groups $2\ (C_2)$, $3\ (C_3)$, $4\ (C_4)$ and $6\ (C_6)$ and in some cases $\bar{4}\ (S_4)$. This is also the procedure of the direct method. Later on we may consider other groups.

3. Notation

From now on the coordinates will be denoted by x, y, z. A component will be written:

$$t_{x,y,z} \quad \text{or} \quad x, y, z$$

if there is no symmetry in the indices. A symmetric tensor of the second order may be written:

$$t_{xy} \quad \text{or} \quad xy;$$

this tensor transforms as the product $\mathbf{r}^2$ instead of $\mathbf{r}\mathbf{r}_1$ for a non-symmetric tensor.

The piezoelectric tensor, which is symmetric in two indices, may be written:

$$t_{xy,x} \quad \text{or} \quad xy, x$$

and this component transforms as the product

$$xyx_1$$

belonging to two vectors $\mathbf{r}$, $\mathbf{r}_1$.

The elastic tensor is now written as

$$(xy, xz) \; ,$$

the brackets indicating that the tensor is symmetric in the first and the second pair of indices. The transformation is given by the product

$$xyx_1z_1 + x_1y_1xz \; .$$

It is, however, simpler to consider only the first term, belonging to xy, xz. Afterwards the components xy, xz and xz, xy may be replaced by one component, being equal for the elastic tensor.

4. The construction of invariant tensors

Let us begin with the components xx, yy, xy of a symmetric second order tensor, which is to be invariant with respect to an n-tuple axis along the Z axis. The symmetry operations are given by

$$\begin{aligned} x' &= x \cos k\varphi - y \sin k\varphi \\ y' &= x \sin k\varphi + y \cos k\varphi \end{aligned} \qquad (2)$$

where $\varphi = 2\pi/n$, and k ranges from 0 to $n-1$.

Then the transformation of the components is the same as for the products

$$\begin{aligned} xx' &= xx \cos^2 k\varphi - 2xy \cos k\varphi \sin k\varphi + yy \sin^2 k\varphi \\ yy' &= xx \sin^2 k\varphi + 2xy \cos k\varphi \sin k\varphi + yy \cos^2 k\varphi \\ xy' &= xx \cos k\varphi \sin k\varphi + xy\,(\cos^2 k\varphi - \sin^2 k\varphi) \\ &\quad - yy \cos k\varphi \sin k\varphi \; . \end{aligned} \qquad (3)$$

The invariant tensor components are found by summing over k. Higher order tensors will lead to sums of higher powers of the sine and cosine functions. If these sums are calculated beforehand, the construction is rather easy. The results of these calculations are given in Table 1, and the method of evaluation in the Appendix.

With regard to Table 1 it should be noted that all sums containing odd powers of the sine are zero, and also that

$$\sum_k \cos k\varphi = 0 \; .$$

The blanks in the columns towards the right hand side mean that the result is the same as for the general value of n. If the results are different, they are noted under the appropriate heading. We may remark that the columns $n = 2, 3, 4$ of 6 are best calculated by direct summation.

Table 1. *Components of the invariant tensors*

$\varphi = 2\pi/n$	n (general)	$n=2$	$n=3$	$n=4$	$n=6$
$\Sigma \cos^2 k\varphi$	$\frac{1}{2}n$	2			
$\Sigma \sin^2 k\varphi$	$\frac{1}{2}n$	0			
$\Sigma \cos^3 k\varphi$	0		$\frac{3}{4}$		
$\Sigma \cos k\varphi \sin^2 k\varphi$	0		$-\frac{3}{4}$		
$\Sigma \cos^4 k\varphi$	$\frac{3}{8}n$	2		2	
$\Sigma \cos^2 k\varphi \sin^2 k\varphi$	$\frac{1}{8}n$	0		0	
$\Sigma \sin^4 k\varphi$	$\frac{3}{8}n$	0		2	
$\Sigma \cos^5 k\varphi$	0		$\frac{15}{16}$		
$\Sigma \cos^3 k\varphi \sin^2 k\varphi$	0		$-\frac{3}{16}$		
$\Sigma \cos k\varphi \sin^4 k\varphi$	0		$-\frac{9}{16}$		
$\Sigma \cos^6 k\varphi$	$\frac{5}{16}n$	2	$\frac{1}{2}\,\frac{33}{16}$	2	$\frac{33}{16}$
$\Sigma \cos^4 k\varphi \sin^2 k\varphi$	$\frac{1}{16}n$	0	$\frac{1}{2}\cdot\frac{3}{16}$	0	$\frac{3}{16}$
$\Sigma \cos^2 k\varphi \sin^4 k\varphi$	$\frac{1}{16}n$	0	$\frac{1}{2}\cdot\frac{9}{16}$	0	$\frac{9}{16}$
$\Sigma \sin^6 k\varphi$	$\frac{5}{16}n$	0	$\frac{1}{2}\cdot\frac{27}{16}$	2	$\frac{27}{16}$

The general derivation is useful for devising those values of n where different sums are found.

If this table is applied to the expressions (3) we find (omitting the factor n)

$$T_{xx}=\tfrac{1}{2}(t_{xx}+t_{yy})$$
$$T_{yy}=\tfrac{1}{2}(t_{xx}+t_{yy})=T_{xx}$$
$$T_{xy}=0;$$

only for $n=2$ we find no relations between the three components.

As n is arbitrary we may put also $n=\infty$; this gives the tensor invariant for a space of cylindrical symmetry. Thus, the blanks in the last four columns indicate that the tensor components constructed with those sums possess the same properties as those in a cylindrical space.

5. Some examples

It will be sufficient to study the elasticity tensor (xx, xx). The 21 components will be considered separately after having been classified according to the number of z's appearing in the components.

(1) (zz, zz) is invariant and (xz, zz) and (yz, zz) are zero.

(2) (xx, zz), (yy, zz) and (xy, zz) have the same properties as the symmetric tensor of the second order. The same is true for (xz, xz), (yz, yz) and (xz, yz). With exception of class 2 (C_2) we find that the first two components of each set are equal, and the third is zero.

(3) (xx, xz), (xx, yz), (yy, xz), (yy, yz), (xy, xz), (xy, yz) are transformed as the corresponding components of the third-order piezoelectric tensor. This transformation is found with help of the products $x^2x_1z_1$ *etc.* The component z is invariant; thus the transformation of x^2x_1 will be sufficient. As all transformed components possess products of the third degree, they must be zero, as may be seen from Table 1, except for $n=3$. In this case we proceed as follows.

$$(x^2x_1z_1)(k\varphi)=(x^2\cos^2 k\varphi-2xy\cos k\varphi\sin k\varphi + y^2\sin^2 k\varphi)(x_1\cos k\varphi-y_1\sin k\varphi)z_1\,.$$

When working out this expression, we need retain only the terms with even powers of the sine. Thus x^2 has only to be multiplied by x_1 *etc.* The result is

$$T_{(xx,xz)}=\tfrac{3}{4}t_{(xx,xz)}-2\,.\,\tfrac{3}{4}t_{(xy,yz)}-\tfrac{3}{4}t_{(yy,xz)}\,.$$

From

$$(xyy_1z_1)(k\varphi)=[x^2\cos k\varphi\sin k\varphi+xy\,(\cos^2 k\varphi-\sin^2 k\varphi) - y^2\cos k\varphi\sin k\varphi](x_1\sin k\varphi+y_1\cos k\varphi)z_1$$

we find

$$T_{(xy,yz)}=-\tfrac{3}{4}t_{(xx,xz)}+2\,.\,\tfrac{3}{4}t_{(xy,yz)}+\tfrac{3}{4}t_{(yy,xz)}=-T_{(xx,xz)}\,.$$

The final result is

$$T_{(xx,xz)}=-T_{(xy,yz)}=-T_{(yy,xz)}$$
$$T_{(yy,yz)}=-T_{(xy,xz)}=-T_{(xx,yz)}\,.$$

(4) (xx, xx), (xx, yy), (yy, yy), (xx, xy), (yy, xy), (xy, xy) will contain in their transforms terms of the fourth degree in the goniometric functions. Their properties will be the same for all axes, except $n=2$ or 4. The general case will exhibit a new feature.

For (xx, xx) we find, omitting the factor n

$$T_{(xx,xx)}=\tfrac{3}{8}t_{(xx,xx)}+\tfrac{1}{8}t_{(xx,yy)}+\tfrac{1}{2}t_{(xy,xy)} + \tfrac{1}{8}t_{(yy,xx)}+\tfrac{3}{8}t_{(yy,yy)}$$

from the product $x^2x_1^2$.

Now $t_{(xx,yy)}=t_{(yy,xx)}$. Moreover we shall find in all equations the same factors for (xx, xx) and for (yy, yy), and we put

$$t_{(xx,xx)}+t_{(yy,yy)}=2t'_{(xx,xx)}\,.$$

For $T_{(yy,yy)}$ we find the same expression; thus

$$T_{(xx,xx)}=T_{(yy,yy)}=\tfrac{3}{4}t'_{(xx,xx)}+\tfrac{1}{4}t_{(xx,yy)}+\tfrac{1}{2}t_{(xy,xy)}\,.$$

In the same way we derive

$$T_{(xx,yy)}=\tfrac{1}{4}t'_{(xx,xx)}+\tfrac{3}{4}t_{(xx,yy)}-\tfrac{1}{2}t_{(xy,xy)}$$
$$T_{(xy,xy)}=\tfrac{1}{4}t'_{(xx,xx)}-\tfrac{1}{4}t_{(xx,yy)}+\tfrac{1}{2}t_{(xy,xy)}\,.$$

There would not be any relation between the three components, if these equations were independent. But the determinant is zero. If we still try to solve t' from these equations in $T_{(xx,xx)}$, $T_{(xx,yy)}$ and $T_{(xy,xy)}$ we find in the numerator

$$\tfrac{1}{4}T_{(xx,xx)}-\tfrac{1}{4}T_{(xx,yy)}-\tfrac{1}{2}T_{(xy,xy)}\,,$$

and as the denominator is zero, being the determinant, the form given above must be zero too, yielding the well known relation between the three components.

The same method may be applied to higher-order tensors, *e.g.* (xx, xx, xx), but this can be better postponed to part II.

6. Other classes

The evaluation of tensors which are invariant with respect to $\bar{4}$ (S_4) does not differ from that for 4 (C_4) if the number of z's is even. If a component contains an odd number of z's, the sum $\sum_k (-1)^k\cos^p k\varphi\sin^r k\varphi$ has to be evaluated. This is readily done for this simple rotation. The method does not differ from the direct method for other classes. It will suffice to give two examples. The class 32 (D_3) has a group of 6 elements, which may be written as the product of three elements, belonging to a ternary axis, u, and two elements v, belonging to a diad axis along the X axis, $s=uv$.

The elements u are the identity, the rotation of 120° and of 240°, whereas v stands for the identity and the rotation of 180°.

We evaluate

$$\mathbf{T}'=\sum_u \mathbf{t}(u)$$

which is the invariant tensor with respect to 3.

Then the tensor $\mathbf{T}$ is found by

$$\mathbf{T}=\sum_v \mathbf{T}'(v)=\Sigma\mathbf{t}(vu)$$

which means in this case that all components, for which the number of y and z together is odd, are cancelled. Of course the summation over v may be done first, as vu and uv give the same elements of 32.

From the class 222 we pass to 23 (T) by multiplying with the elements of a ternary axis along the body diagonal:

$$x'=x,\ y'=y,\ z'=z$$
$$x'=y,\ y'=z,\ z'=x$$
$$x'=z,\ y'=x,\ z'=y\ ,$$

which means that the tensor is found from the tensor for 222 by summing a component with its transformed ones. These are found by cyclic permutation of the indices x, y and z.

II. THE MODIFIED FUMI METHOD

7. Description of the method

The method described above constructs linear combinations of tensor components which are invariant for the group we consider.

Now we may find other combinations, independent of the invariant ones, and these combinations must be zero. This is the method of Fumi, but we shall derive these combinations in another way.

We need only to consider coordinates x and y, and the cyclic groups. For other classes the method explained in § 6 will serve.

Theorem

The tensor components can be combined into pairs of linear combinations, which, considered as vectors, rotate over an angle $m\varphi$, $(m-2)\varphi$, ..., φ or 0, if the space xy is rotated over an angle φ, and the order of the tensor in x and y is m. We will refer to them as m-vectors. A 0-vector is an invariant for each rotation, and may appear single.

The proof follows from the method of construction.

The second order tensor has the components $x, x;\ x, y;\ y, x;\ y, y$.

We write

$$x=r\cos\psi,\ y=r\sin\psi$$

then a rotation means that ψ is replaced by $\psi+\varphi$.

The second order tensor is transformed as the product xx_1 *etc.*

It follows that

$$xx_1+yy_1=rr_1\cos(\psi-\psi_1)$$
$$xx_1-yy_1=rr_1\cos(\psi+\psi_1)$$
$$xy_1+yx_1=rr_1\sin(\psi+\psi_1)$$
$$xy_1-yx_1=rr_1\sin(\psi_1-\psi)\ .$$

Rotation means that φ is added to ψ and to ψ_1; thus the first and last combination are invariants, the second and third form a 2-vector. These are zero for an arbitrary rotation, and we find for the tensor components:

$$x, x=y, y,\quad x, y=-y, x\ .$$

But this is not true if $\varphi=180°$. Then the double rotation is equivalent to the identity, the four combinations are invariants, and no relation exists between the tensor components. We remark that the factors r, r_1 are not necessary, and they will be left out.

The symmetric tensor leads to

$$x^2+y^2=1$$
$$x^2-y^2=\cos 2\psi$$
$$2xy\quad=\sin 2\psi\ .$$

Except for $\varphi=180°$, the 2-vector must be zero, which yields:

$$xx=yy,\ xy=0\ .$$

We turn now to the piezoelectric tensor xx, x, which is transformed as the product x^2x_1. We study the products of the invariant x^2+y^2 and the 2-vector x^2-y^2, $2xy$ with the vector x_1, y_1. The result is

$$(x^2+y^2)x_1=\cos\psi_1$$
$$(x^2+y^2)y_1=\sin\psi_1$$
$$(x^2-y^2)x_1-2xyy_1=\cos(2\psi+\psi_1)$$
$$2xyx_1+(x^2-y^2)y_1=\sin(2\psi+\psi_1)$$
$$(x^2-y^2)x_1+2xyy_1=\cos(2\psi-\psi_1)$$
$$2xyx_1-(x^2-y^2)y_1=\sin(2\psi-\psi_1)$$

with two vectors and one 3-vector. The last four are found in the same way as above, the vector x, y being replaced by the 2-vector: $\cos 2\psi$, $\sin 2\psi$. In the general case all combinations must be zero, and all tensor components are zero too. But for $n=3$ the 3-vector is an invariant, and we are left with

$$xx, x+yy, x=0,\quad xx, y+yy, y=0$$
$$xx, x-yy, x+2xy, y=0$$
$$2xy, x-xx, y+yy, y=0$$

from which the well known properties result.

Our last example will be the symmetric third-order tensor with four components, *viz.*

$$xxx \text{ as } x^3=\cos^3\psi$$
$$xxy \text{ as } x^2y=\cos^2\psi\sin\psi$$
$$xyy \text{ as } xy^2=\cos\psi\sin^2\psi$$
$$yyy \text{ as } y^3=\sin^3\psi\ .$$

From

$$\cos 3\psi=\cos^3\psi-3\cos\psi\sin^2\psi$$
$$\sin 3\psi=3\cos^2\psi\sin\psi-\sin^3\psi$$

we find the 3-vector $xxx-3xyy$, $3xxy-yyy$.

The components of the vector are

$$xxx+xyy \quad\text{and}\quad xxy+yyy$$

and for $n=3$ only these combinations are zero.

8. A sixth-order tensor

We shall apply the method to the tensor (xx, xx, xx) which is symmetric in the three pairs xx. As before, we classify the components according to the number of z's. It is seen at once that (zz, zz, zz) is invariant and

$$(xz, zz, zz) = (yz, zz, zz) = 0 .$$

Further

I(*a*) (xx, zz, zz), (yy, zz, zz), (xy, zz, zz) have the same properties as xx; yy; xy.

I(*b*) The same is true for (xz, xz, zz), (yz, yz, zz) and (xz, yz, zz).

II(*a*) (xx, xz, zz), (xx, yz, zz), (yy, xz, zz), (yy, yz, zz), (xy, xz, zz) and (xy, yz, zz) transform as xx, x *etc.*, the piezoelectric tensor.

II(*b*) (xz, xz, xz), (xz, xz, yz), (xz, yz, yz), (yz, yz, yz) transform as (xxx) which we considered in § 7.

III(*a*) (xx, xx, zz), (xx, yy, zz), (yy, yy, zz), (xx, xy, zz), (yy, xy, zz) and (xy, xy, zz) may be replaced by the elastic tensor (xx, xx).

III(*b*) (xx, xz, xz), (xx, yz, yz), (yy, xz, xz), (yy, yz, yz), (xx, xz, yz), (xy, xz, xz), (yy, xz, yz), (xy, yz, yz) and (xy, xz, yz) represent the tensor (xx, x, x), which may be treated by both methods.

The result is for the general case, including $n=3$ or 6:

$(xx, xz, yz) = (xy, xz, xz) = (yy, xz, yz) = (xy, yz, yz) = 0$
$(xx, xz, xz) = (yy, yz, yz)$
$(xx, yz, yz) = (yy, xz, xz)$
$(xx, xz, xz) - (xx, yz, yz) = 2(xy, xz, yz)$.

This is the same as III(*a*), if we replace (xx, yz, yz) and (yy, xz, xz) by (xx, yy, zz) *etc.*

IV (xx, xx, xz), (xx, xx, yz), (xx, yy, xz), (xx, yy, yz), (yy, yy, xz), (yy, yy, yz), (xx, xy, xz), (xx, xy, yz), (yy, xy, xz), (yy, xy, yz), (xy, xy, xz), (xy, xy, yz). These components will be zero, in virtue of Table 1. Only for $n=3$ (and $n=5$) this will not be the case. The Fumi method is better suited to the study of this tensor. The same is true for:

V (xx, xx, xx), (xx, xx, yy), (xx, yy, yy), (yy, yy, yy), (xx, xx, xy), (xx, yy, xy), (yy, yy, xy), (xx, xy, xy), (yy, xy, xy), (xy, xy, xy).

For V we may proceed as follows: The pairs xx; yy; xy are replaced by the invariant x^2+y^2, and the 2-vector $x^2-y^2 = \cos 2\psi$, $2xy = \sin 2\psi$. In the evaluation of the m-vectors, ψ_1 and ψ_2 may be put equal to ψ.

(*a*) $(x^2+y^2)(x^2+y^2)(x^2+y^2) = 1$

(*b*) $(x^2+y^2)(x^2+y^2)(x^2-y^2) = \cos 2\psi$
$(x^2+y^2)(x^2+y^2)2xy \quad = \sin 2\psi$.

(*c*) From two 2-vectors we find an invariant and a 4-vector, and they remain so after having been multiplied by an invariant x^2+y^2:

$$(x^2+y^2)[(x^2-y^2)(x^2-y^2)+(2xy)(2xy)] = 1$$
$$(x^2+y^2)[(x^2-y^2)(x^2-y^2)-(2xy)(2xy)] = \cos 4\psi$$
$$(x^2+y^2)[2(x^2-y^2)(2xy)] = \sin 4\psi.$$

(*d*) Here we find the product of three 2-vectors. In the last paragraph the product of three vectors was studied; we use here the same method. From the 2-vector and the 6-vector we note only the first one

$$(x^2-y^2)(x^2-y^2)(x^2-y^2)+(x^2-y^2)(2xy)(2xy) = \cos 2\psi$$
$$(x^2-y^2)(x^2-y^2)(2xy)+(2xy)(2xy)(2xy) = \sin 2\psi.$$

For $n=3$ or 6 all combinations are zero, excepted the invariants and the 6-vector. For a 6-vector is rotated over 6φ, and $n=3$ means $\varphi = 120°$. An m-vector is an invariant for an n-tuple axis, if n is a factor of m. The products are easily translated into tensor components, only it should be remembered that $(xx, yy, xx) = (xx, xx, yy)$ *etc.* For (*c*) we find

$$(xx, xx, xx) - (xx, xx, yy) - (xx, yy, yy) + (yy, yy, yy) - 4(xx, xy, xy) - 4(xx, xy, xy) = 0$$
$$(xx, xx, xy) - (yy, yy, xy) = 0.$$

The equations may be rearranged as follows:

$$(xx, xx, xx) + (xx, xx, yy) = (xx, yy, yy) + (yy, yy, yy)$$
$$(xx, xy, xy) + (yy, xy, xy) = \tfrac{1}{2}(xx, xx, xx) - \tfrac{1}{2}(xx, yy, yy)$$
$$(xx, xy, xy) - (yy, xy, xy) = -(xx, xx, xx) + (yy, yy, yy)$$
$$(xx, xx, xy) = (yy, yy, xy) = -(xx, yy, xy) = -(xy, xy, xy).$$

We turn now to IV. We borrow from (*a*), (*b*) and (*c*) the invariants, the 2-vector and 4-vector, belonging to (xx, xx) and combine them with the vector x, y, instead of the invariant x^2+y^2. Putting all combinations zero, excepted the 3-vectors, we find for $n=3$, after rearranging, the equations:

$$(xx, yy, xz) =$$
$$(xy, xy, xz) = -\tfrac{1}{2}(xx, xx, xz) - \tfrac{1}{2}(yy, yy, xz)$$
$$(xx, xy, yz) = \tfrac{1}{4}(xx, xx, xz) + \tfrac{3}{4}(yy, yy, xz)$$
$$(yy, xy, yz) = -\tfrac{3}{4}(xx, xx, xz) - \tfrac{1}{4}(yy, yy, xz)$$

$$(xx, yy, yz) =$$
$$(xy, xy, yz) = -\tfrac{1}{2}(xx, xx, yz) - \tfrac{1}{2}(yy, yy, yz)$$
$$(xx, xy, xz) = -\tfrac{1}{4}(xx, xx, yz) - \tfrac{3}{4}(yy, yy, yz)$$
$$(yy, xy, xz) = \tfrac{3}{4}(xx, xx, yz) + \tfrac{1}{4}(yy, yy, yz).$$

The second set of equations may be found from the first set by permutation of x and y.

We may add that the number of linear combinations always equals the number of components, as follows from their construction. It is not necessary to look for combinations which might be dependent on the others. As an example the last case (IV) may serve. The two invariants give two vectors, after having been combined with the vector x, y. The 2-vector leads to a vector and a 3-vector, the 4-vector to a 3-vector and a 5-vector. There are 6 vectors and 12 combinations for 12 components. The three vectors and the 5-vector, being zero, give 8 equations. These are solved in (xx, xx, xz), (yy, yy, xz), (xx, xx, yz) and (yy, yy, yz), yielding the 8 relations given above.

APPENDIX

The evaluation of the sums

$$\sum_0^{n-1} \cos^p k\varphi \sin^r k\varphi, \quad \varphi = 2\pi/n ,$$

which constitute Table 1, will be given here only for $p=r=2$. As

$$\begin{aligned} \cos k\varphi &= \tfrac{1}{2}[\exp(ik\varphi)+\exp(-ik\varphi)] \\ -\sin k\varphi &= \tfrac{1}{2}i[\exp(ik\varphi)-\exp(-ik\varphi)] \\ \cos^2 k\varphi \sin^2 k\varphi &= -\tfrac{1}{16}[\exp(4ik\varphi)+\exp(-4ik\varphi)-2]. \end{aligned}$$

By virtue of

$$\sum_0^{n-1} \exp 4ik\varphi = \frac{1-\exp 4in\varphi}{1-\exp 4i\varphi} = 0$$

we find

$$\sum_j^{n-1} \cos^2 k\varphi \sin^2 k\varphi = n/8 .$$

But for $n=4$ not only is $\exp 4in\varphi = 1$, but also $\exp 4i\varphi = 1$. Then the derivation breaks down, and the sum is better calculated directly. In this way the results in Table 1 are found.

The expressions derived above may be of use too for the modified Fumi method. Whereas the composition of two vectors is simple, the case (xxx), where three vectors had to be combined in § 7, was solved by trying to find expressions for $\cos 3\psi$ *etc.* That this can be done systematically will be shown for $(xxxx)$. Here we meet $\cos^4\psi$, $\cos^3\psi\sin\psi$, $\cos^2\psi\sin^2\psi$, *etc.* The expression we found for the last product may be written:

$$\cos^2\psi \sin^2\psi = -\tfrac{1}{8}\cos 4\psi + \tfrac{1}{8} .$$

In the same way we find

$$\begin{aligned} \cos^4\psi &= \tfrac{1}{8}\cos 4\psi + \tfrac{1}{2}\cos 2\psi + \tfrac{3}{8} \\ \sin^4 \quad &= \tfrac{1}{8}\cos 4\psi - \tfrac{1}{2}\cos 2\psi + \tfrac{3}{8} . \end{aligned}$$

These may be considered as three equations in the variables $\cos 4\psi$, $\cos 2\psi$, 1. Solving them we find

$$\begin{aligned} \cos 4\psi &= \cos^4\psi + \sin^4\psi - 6\cos^2\psi\sin^2\psi \\ \cos 2\psi &= \cos^4\psi - \sin^4\psi , \end{aligned}$$

from which a component of a 4-vector and of a 2-vector is found, *viz.*

$$\begin{aligned} &(xxxx)+(yyyy)-6(xxyy) \\ &(xxxx)-(yyyy). \end{aligned}$$

By calculating $\cos^3\psi\sin\psi$ and $\cos\psi\sin^3\psi$, we find

$$\begin{aligned} \sin 4\psi &= 4\,(\cos^3\psi\sin\psi - \cos\psi\sin^3\psi) \\ \sin 2\psi &= 2\,(\cos^3\psi\sin\psi + \cos\psi\sin^3\psi) \end{aligned}$$

which yield the remaining combinations, belonging to the 4-vector and the 2-vector

$$\begin{aligned} &4(xxxy)-4(xyyy) \\ &2(xxxy)+2(x.yyy) . \end{aligned}$$

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